

Title	RESIDUE OF CODIMENSION 1 SINGULAR HOLOMORPHIC DISTRIBUTIONS(Recent Topics on Real and Complex Singularities)
Author(s)	Izawa, Takeshi
Citation	数理解析研究所講究録 (2006), 1501: 124-131
Issue Date	2006-07
URL	http://hdl.handle.net/2433/58425
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

RESIDUE OF CODIMENSION 1 SINGULAR HOLOMORPHIC DISTRIBUTIONS

伊澤 毅 (Takeshi Izawa) 北海道大学 理

1

The aim of this note is to describe the residue formula for singular holomorphic distribution in terms of the conormal sheaf \mathcal{G} in codimension 1 case.

We also prove the Baum-Bott type residue formula for singular distributions. If we define the tangent sheaf of the distribution \mathcal{F} by taking the annihilator of \mathcal{G} by the dual coupling, we will show that the residue formula for \mathcal{G} deduce the Baum-Bott type residue formula for the top Chern class of the normal sheaf $\mathcal{N}_{\mathcal{F}}$. If we assume the Frobenius integrability condition for \mathcal{G} , we have the Baum-Bott residue formula

$$\int_X \varphi(\mathcal{N}_{\mathcal{F}}) = \text{Res}_{\varphi}(\mathcal{N}_{\mathcal{F}}, S(\mathcal{F}))$$

for n -th symmetric polynomial φ . In this case, the Baum-Bott residue formula for $\varphi = c_n$ is equivalent to the formula we will prove, which means that the Bott vanishing theorem based on the involutivity of \mathcal{F} is not necessary for the top Chern class $c_n(\mathcal{N}_{\mathcal{F}})$.

As an application of our results, we will give a residue formula for the non-transversality of a holomorphic map $F : X \rightarrow Y$ to a non-singular distribution on Y .

2. SINGULAR HOLOMORPHIC DISTRIBUTION

2.1. Singular holomorphic distribution. Let X be a complex manifold. We define a singular holomorphic distribution \mathcal{F} on X to be a coherent subsheaf of the tangent sheaf Θ_X . we call \mathcal{F} the tangent sheaf of the distribution. We say \mathcal{F} is dimension p if a generic stalk of \mathcal{F} is rank p free \mathcal{O}_X -module. We also define the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of \mathcal{F} by the exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \Theta_X \rightarrow \mathcal{N}_{\mathcal{F}} \rightarrow 0.$$

The singular set $S(\mathcal{F})$ of \mathcal{F} is defined by $S(\mathcal{F}) = \{p \in X \mid \mathcal{N}_{\mathcal{F},p} \text{ is not } \mathcal{O}_p\text{-free}\}$.

We can also give a definition of a singular holomorphic distribution \mathcal{G} on X to be a coherent subsheaf of the cotangent sheaf Ω_X . We call \mathcal{G} the conormal sheaf of the distribution. We also say \mathcal{G} is codimension q if the generic rank is q . We also define the cotangent sheaf $\Omega_{\mathcal{G}}$ of \mathcal{G} by the exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \Omega_X \rightarrow \Omega_{\mathcal{G}} \rightarrow 0.$$

The singular set $S(\mathcal{F})$ of \mathcal{F} is also defined by $S(\mathcal{G}) = \{p \in X \mid \Omega_{\mathcal{G},p} \text{ is not } \mathcal{O}_p\text{-free}\}$.

2.2. Codimension 1 case. we give more simple descriptions for codimension 1 singular distributions. A codimension 1 locally free singular holomorphic distribution is given by a collection of 1-forms $\omega = (\omega_\alpha, U_\alpha)$ for an open covering $\{U_\alpha\}$ of X which has the transition relations $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on the intersection $U_\alpha \cap U_\beta$ with $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Then the cocycle $(g_{\alpha\beta})$ defines a line bundle G . Generically at p , the covector ω_p gives an embedding of the fiber G_p into T_p^*X by $f_p \in G_p \mapsto f_p\omega_p \in T_p^*X$. Thus G is regarded as a subbundle of T^*X without on the zero loci of ω . Since the map of germs of sections $(f)_p \in \mathcal{O}_X(G)_p \mapsto (f\omega)_p \in \Omega_{X,p}$ are injective for all $p \in X$, the sheaf $\mathcal{G} = \mathcal{O}_X(G)$ gives the subsheaf of Ω_X in the above sense in 1.2. Since the quotient sheaf $\Omega_{\mathcal{F}}$ is not \mathcal{O} -free on the zero loci of ω on which we can not define the quotient bundle T^*X/G , we see the singular set of \mathcal{G} is $S(\mathcal{G}) = \{p \mid \omega(p) = 0\}$.

3. RESIDUE OF CODIMENSION 1 DISTRIBUTION

3.1. Localization of the top Chern class. We determine the dual homology class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. Our main tool is the Čech-de Rham techniques. For generalities on the integration and the Chern-Weil theory on the Čech-de Rham cohomology, see [S3] or [IS]. We set for an analytic set S , $U_0 = X \setminus S$, U_1 is a regular neighbourhood of S , and $U_{01} = U_0 \cap U_1$. For a covering $\mathcal{U} = \{U_0, U_1\}$ of X , the Čech-de Rham cohomology group $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ is represented by the group of cocycles of the type $(\sigma_0, \sigma_1, \sigma_{01})$ for $\sigma_0 \in Z^{2n}(U_0)$, $\sigma_1 \in Z^{2n}(U_1)$, and $\sigma_{01} \in A^{2n-1}(U_{01})$ with $d\sigma_{01} = \sigma_1 - \sigma_0$. We note that the Čech-de Rham cohomology can be regarded as the hypercohomology of the de Rham complex $\{\mathcal{A}^\bullet, d\}$. By usual spectral sequence arguments for double complexes, we see that the Čech-de Rham cohomology group is canonically isomorphic to the de Rham cohomology group. If we take the subgroup $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}, U_0))$ of cocycles of the form $(0, \sigma_1, \sigma_{01})$, then this is also isomorphic to the relative cohomology group $H^{2n}(X, X \setminus S; \mathbb{C})$.

In the above settings, the top Chern class $c_n(E)$ of a vector bundle E of rank n is given by the cocycle in $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ as follows. For $i = 0, 1$, let ∇_i be a connection for E on U_i and $c_n(\nabla_i)$ the n -th Chern form of ∇_i . We also write by $c_n(\nabla_0, \nabla_1)$ the transgression form of $c_n(\nabla_i)$'s on U_{01} . Then $c_n(E)$ is represented by

$$(c_n(\nabla_1), c_n(\nabla_1), c_n(\nabla_0, \nabla_1)).$$

If E has a global section s with zero loci S , then we take ∇_0 as the s -trivial connection such that we have $c_n(\nabla_0) = 0$. Thus we can define the localized Chern class at p in $H^{2n}(X, X \setminus S; \mathbb{C})$ by a Čech-de Rham cocycle $(0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1))$.

The integration of $c_n(E) = (0, c_n(\nabla_1), c_n(\nabla_0, \nabla_1))$ is defined by

$$\int_X c_n(E) = \int_R c_n(\nabla_1) - \int_{\partial R} c_n(\nabla_0, \nabla_1)$$

for a tubular neighbourhood $R \subset U_1$ of S .

3.2. Residue of codimension 1 distributions. Now we apply the above arguments to our situations. Let \mathcal{G} be a codimension 1 locally free distribution with the zero loci $S(\mathcal{G})$ and we suppose that $S(\mathcal{G})$ has connected components S_j . We set $U_0 = X \setminus S(\mathcal{G})$ and U_j is a regular neighbourhood of S_j . We consider the localized class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$ in the Čech-de Rham cohomology group for the covering $\mathcal{U} = \{U_0, U_1, \dots, U_j\}$. Since the collection ω of 1-forms ω_α defines the global section of $\Omega_X \otimes \mathcal{G}^\vee$, we can take ∇_0 as the ω -trivial connection such that $c_n(\nabla_0) = 0$

as we discussed above. For all $j = 1, \dots, k$, we can also take ∇_j as an arbitrary connection on U_j . So we have

$$c_n(\Omega_X \otimes \mathcal{G}^\vee) = (0, \{c_n(\nabla_j)\}_{j=1, \dots, k}, \{c_n(\nabla_0, \nabla_j)\}_{j=1, \dots, k}) \in H^{2n}(X, X \setminus S(\mathcal{G}); \mathbb{C}).$$

We denote by R_j a tubular neighbourhood of S_j in U_j . We give the following definition of residue.

Definition 3.1. *The residue of \mathcal{G} at S_j is defined by*

$$\text{Res}(\mathcal{G}, S_j) = \int_{R_j} c_n(\nabla_j) - \int_{\partial R_j} c_n(\nabla_0, \nabla_j).$$

We can describe the residue into precise form in isolated singular cases. Here we refer the result in [S3] of Theorem 5.5.

Theorem 3.2. *Let s be a regular section of E with isolated zero $\{p\}$ and s is locally given by (f_1, \dots, f_n) near p . Then we have*

$$\text{Res}(\mathcal{G}, p) = \text{Res}_p \left[\frac{df_1 \wedge \dots \wedge df_n}{f_1 \dots f_n} \right]$$

where $\text{Res}_p \left[\frac{df_1 \wedge \dots \wedge df_n}{f_1 \dots f_n} \right]$ is the Grothendick residue of (f_1, \dots, f_n) .

The dual correspondence in the Alexander duality

$$AL : H^{2n}(X \setminus S(\mathcal{G}); \mathbb{C}) \xrightarrow{\sim} \bigoplus_j H_0(S_j; \mathbb{C})$$

is given by

$$AL(c_n(\Omega_X \otimes \mathcal{G}^\vee)) = \sum_j \text{Res}(\mathcal{G}, S_j).$$

Now we have the residue formula for isolated singular cases as,

Theorem 3.3 (The residue formula for isolated singularities). *Let ω be a codimension 1 singular holomorphic distribution with the cotangent sheaf \mathcal{G} and $(f_1^{(j)}, \dots, f_n^{(j)})$ a local expression of $\omega \in H^0(X, \Omega_X \otimes \mathcal{G}^\vee)$ near p_j .*

$$\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) = \sum_{j=1}^k \text{Res}_{p_j} \left[\frac{df_1^{(j)} \wedge \dots \wedge df_n^{(j)}}{f_1^{(j)} \dots f_n^{(j)}} \right].$$

4. BAUM-BOTT TYPE RESIDUE FORMULA

4.1. Koszul resolution. First let us remember the definition of the Koszul complex. (See [FG], Chapter 4 or [GH], Chapter 5.) Let \mathcal{E} be a locally free \mathcal{O} -module of rank n and $d : \mathcal{E} \rightarrow \mathcal{O}$ an \mathcal{O} -homomorphism. Then the Koszul complex of sheaves

$$0 \rightarrow \wedge^n \mathcal{E} \rightarrow \wedge^{n-1} \mathcal{E} \rightarrow \dots \rightarrow \wedge^1 \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$$

is defined by the boundary operator

$$d_p(\varepsilon_1 \wedge \dots \wedge \varepsilon_p) = \sum_{i=1}^p (-1)^{i-1} d(\varepsilon_i) \varepsilon_1 \wedge \dots \wedge \hat{\varepsilon}_i \wedge \dots \wedge \varepsilon_p.$$

This complex is exact except for the last term. If the image \mathcal{I}_d of d is regular ideal, the complex

$$0 \rightarrow \wedge^n \mathcal{E} \rightarrow \wedge^{n-1} \mathcal{E} \rightarrow \dots \rightarrow \wedge^1 \mathcal{E} \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}_d \rightarrow 0$$

is exact. We call this exact sequence the Koszul resolution of $\mathcal{O}/\mathcal{I}_d$.

Now let us consider our case. As observed in 2.1, ω can be regarded as a homomorphism $\omega : \mathcal{G} \rightarrow \Omega_X$ such that it defines a global section

$$\omega \in H^0(X, \text{Hom}_{\mathcal{O}}(\mathcal{G}, \Omega_X)) \simeq H^0(X, \Omega_X \otimes \mathcal{G}^\vee).$$

Locally on U_α , ω is given by $\omega_\alpha \otimes s_\alpha^\vee = \sum f_i(dx_i \otimes s_\alpha^\vee)$ for some local coordinates of X and a local frame s_α^\vee for \mathcal{G}^\vee . In the other words, ω acts on $(\Omega_X \otimes \mathcal{G}^\vee)^\vee \simeq \Theta_X \otimes \mathcal{G}$ as a contraction operator $\omega : \Theta_X \otimes \mathcal{G} \rightarrow \mathcal{O}$. We denote by \mathcal{I}_ω the ideal sheaf defined by $\text{Im}(\omega : \Theta_X \otimes \mathcal{G} \rightarrow \mathcal{O})$. We assume that $S(\mathcal{G}) = \{p \in X \mid \omega_p = 0\}$ consists only of isolated points such that the local coefficients (f_1, \dots, f_n) of ω is regular sequence on $S(\mathcal{G})$. Then the complex of sheaves

$$0 \rightarrow \wedge^n(\Theta_X \otimes \mathcal{G}) \rightarrow \wedge^{n-1}(\Theta_X \otimes \mathcal{G}) \rightarrow \dots \rightarrow \wedge^1(\Theta_X \otimes \mathcal{G}) \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}_\omega \rightarrow 0$$

is exact with the boundary operator

$$d_p(e_1 \wedge \dots \wedge e_p) = \sum_{i=1}^p (-1)^{i-1} f_i e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_p$$

where we set $e_i = \frac{\partial}{\partial x_i} \otimes s$. Therefore this gives the Koszul resolution of $\mathcal{O}/\mathcal{I}_\omega$.

By using this projective resolution, we can define the Chern character of the coherent sheaf $\mathcal{O}/\mathcal{I}_\omega$ by

Proposition 4.1.

$$ch(\mathcal{O}/\mathcal{I}_\omega) = c_n(\Omega_X \otimes \mathcal{G}^\vee).$$

Proof. We use [H] of Theorem 10.1.1 and we have

$$\begin{aligned} ch(\mathcal{O}/\mathcal{I}_\omega) &= ch\left(\sum_{i=0}^n (-1)^i \wedge^i(\Theta_X \otimes \mathcal{G})\right) \\ &= td^{-1}(\Omega_X \otimes \mathcal{G}^\vee) c_n(\Omega_X \otimes \mathcal{G}^\vee) \\ &= c_n(\Omega_X \otimes \mathcal{G}^\vee). \end{aligned}$$

4.2. Baum-Bott type residue formula. Now we translate the above results in terms of differential system in the tangent sheaf Θ_X : Let $\mathcal{F} = \{v \in \Theta_X \mid \langle v, \omega \rangle = 0\}$ be the annihilator of \mathcal{G} . Then \mathcal{F} defines a $n-1$ dimensional (possibly) singular distribution. Since \mathcal{G} is locally free, by applying $\otimes \mathcal{G}$ to the exact sequence

$$(1) \quad 0 \rightarrow \mathcal{F} \rightarrow \Theta_X \rightarrow \mathcal{N}_{\mathcal{F}} \rightarrow 0,$$

the following sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{G} \rightarrow \Theta_X \otimes \mathcal{G} \rightarrow \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G} \rightarrow 0.$$

is also exact. Since the kernel of $\omega : \Theta_X \otimes \mathcal{G} \rightarrow \mathcal{O}_X$ is equals to $\mathcal{F} \otimes \mathcal{G}$, we have

$$(2) \quad \mathcal{I}_\omega \simeq (\Theta_X \otimes \mathcal{G}) / (\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G}.$$

We take $\text{Hom}_{\mathcal{O}}(_, \mathcal{O})$ of the dual exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \Omega_X \rightarrow \Omega_{\mathcal{G}} \rightarrow 0$$

of (1), we obtain the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \rightarrow \text{Hom}_{\mathcal{O}}(\Omega_X, \mathcal{O}) \rightarrow \text{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{O}) \rightarrow \text{Ext}_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O}) \rightarrow 0,$$

which implies

$$0 \rightarrow \mathcal{F} \rightarrow \Theta_X \rightarrow \mathcal{G}^\vee \rightarrow \text{Ext}_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O}) \rightarrow 0.$$

We use $\mathcal{F} = \mathcal{H}om_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})$ and $\Theta_X = \mathcal{H}om_{\mathcal{O}}(\Omega_X, \mathcal{O})$ in the above. Thus we obtain

$$(3) \quad 0 \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow \mathcal{G}^{\vee} \longrightarrow \mathcal{E}xt_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow 0.$$

By taking the Chern characters of (3), we have

$$(4) \quad ch(\mathcal{N}_{\mathcal{F}}) = ch(\mathcal{G}^{\vee}) - ch(\mathcal{E}xt_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O})).$$

By tensoring \mathcal{G} for each term of (3), we also have the exact sequence

$$0 \longrightarrow \mathcal{I}_{\omega} \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}xt_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G} \longrightarrow 0$$

and which gives the isomorphism $\mathcal{O}/\mathcal{I}_{\omega} \simeq \mathcal{E}xt_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G}$. Thus the Chern characters of those sheaves satisfy

$$(5) \quad ch(\mathcal{E}xt_{\mathcal{O}}^1(\Omega_{\mathcal{G}}, \mathcal{O})) = ch(\mathcal{O}/\mathcal{I}_{\omega})ch(\mathcal{G}^{\vee}).$$

Therefore by combining the two equalities (4) and (5) for the Chern characters, we obtain

Proposition 4.2.

$$\begin{aligned} ch(\mathcal{N}_{\mathcal{F}}) &= (1 - ch(\mathcal{O}/\mathcal{I}_{\omega}))ch(\mathcal{G}^{\vee}) \\ &= (1 - c_n(\Omega_X \otimes \mathcal{G}^{\vee}))ch(\mathcal{G}^{\vee}). \end{aligned}$$

Now we find the top Chern class of $\mathcal{N}_{\mathcal{F}}$.

Proposition 4.3.

$$c_n(\mathcal{N}_{\mathcal{F}}) = (-1)^n(n-1)!c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

Proof. Let $\{\xi_i\}$ be the formal Chern roots of $c(\mathcal{N}_{\mathcal{F}})$ and ch_i the terms of i -th degree in ch . Then from proposition 3.1, we have

$$ch_i(\mathcal{N}_{\mathcal{F}}) = \frac{1}{i!}c_1(\mathcal{G}^{\vee})^i$$

for $i \leq n-1$ and

$$ch_n(\mathcal{N}_{\mathcal{F}}) = \frac{1}{n!}c_1(\mathcal{G}^{\vee})^n - c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

$ch_1(\mathcal{N}_{\mathcal{F}}) = c_1(\mathcal{G}^{\vee})$ is obvious. we also see that

$$\begin{aligned} \frac{1}{2!}c_1(\mathcal{G}^{\vee})^2 &= ch_2(\mathcal{N}_{\mathcal{F}}) \\ &= \frac{1}{2!}(\xi_1^2 + \cdots + \xi_n^2) \\ &= \frac{1}{2!}\{(\xi_1 + \cdots + \xi_n)^2 - 2 \sum \xi_i \xi_j\} \\ &= \frac{1}{2!}c_1(\mathcal{G}^{\vee})^2 - c_2(\mathcal{N}_{\mathcal{F}}), \end{aligned}$$

which implies $c_2(\mathcal{N}_{\mathcal{F}}) = 0$. We continue the same computations for fundamental symmetric polynomials, we have

$$c_2(\mathcal{N}_{\mathcal{F}}) = \cdots = c_{n-1}(\mathcal{N}_{\mathcal{F}}) = 0.$$

Thus for n -th term, we have

$$\begin{aligned}
 \frac{1}{n!} c_1(\mathcal{G}^\vee)^n - c_n(\Omega_X \otimes \mathcal{G}^\vee) &= ch_n(\mathcal{N}_{\mathcal{F}}) \\
 &= \frac{1}{n!} (\xi_1^n + \cdots + \xi_n^n) \\
 &= \frac{1}{n!} \{ (\xi_1 + \cdots + \xi_n)^n - (-1)^n n \xi_1 \cdots \xi_n \} \\
 &= \frac{1}{n!} c_1(\mathcal{G}^\vee)^n - \frac{(-1)^n}{(n-1)!} c_n(\mathcal{N}_{\mathcal{F}}),
 \end{aligned}$$

from which the result follows.

We combin the results in (2.3), we can derive the formula for the normal sheaf $\mathcal{N}_{\mathcal{F}}$, which is the Baum-Bott type residue formula.

Theorem 4.4 (Baum-Bott type residue formula). *Let ω be a codimension 1 distribution with conormal sheaf \mathcal{G} , and \mathcal{F} the annihilator of \mathcal{G} . We suppose that $S(\mathcal{G}) = \{p_1, \dots, p_k\}$ and we write $\omega = \sum f_i^{(j)}(dx_i \otimes s^\vee)$ near p_j . Then we have*

$$\int_X c_n(\mathcal{N}_{\mathcal{F}}) = (-1)^n (n-1)! \sum_j \text{Res} \left[\frac{df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}}{f_1^{(j)} \cdots f_n^{(j)}} \right].$$

proof. This is simply given by

$$\begin{aligned}
 \int_X c_n(\mathcal{N}_{\mathcal{F}}) &= (-1)^n (n-1)! \int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) \\
 &= (-1)^n (n-1)! \sum_j \text{Res} \left[\frac{df_1^{(j)} \wedge \cdots \wedge df_n^{(j)}}{f_1^{(j)} \cdots f_n^{(j)}} \right]
 \end{aligned}$$

Remarks. If we assume the integrability condition on \mathcal{G} , the above formula implies the Baum-Bott residue formula for singular holomorphic foliations. Since the Baum-Bott residue for $c_n(\mathcal{N}_{\mathcal{F}})$ is given by

$$(-1)^n (n-1)! \dim \text{Ext}_{\mathcal{O}_p}^1(\Omega_{\mathcal{G},p}, \mathcal{O}_p) = (-1)^n (n-1)! \dim \mathcal{O}_p / \mathcal{I}_{\omega,p},$$

the right hand side of 3.4 coincides the Baum-Bott residue.

5. APPLICATIONS

5.1. Residue for the non-transversal loci of a holomorphic map. Let $F : X^n \rightarrow Y^m$ be a holomorphic map between n and m dimensional compact complex manifolds. If Y has a non-singular distribution $\tilde{\mathcal{G}} = \mathcal{O}_Y(G)$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ gives a distribution of X which is possibly singular. In codimension 1 case, if a distribution $\tilde{\mathcal{G}}$ on Y is given by a collection of 1-forms $\tilde{\omega} = (\tilde{\omega}_\alpha)$, then the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ of the invertible sheaf $\tilde{\mathcal{G}}$ is given by the collection of 1-forms $\omega = (F^*\tilde{\omega}_\alpha)$. If the image of the differential DF_p dose not contain the normal space G_p^* , we see that covector ω_p is zero. Thus the non-transversal loci of F to $\tilde{\mathcal{G}}$ is given by

$$S(\mathcal{G}) = \{p \in X : F^*\tilde{\omega}_\alpha(p) = 0\}$$

Now we give the residue formula for the non-transversality of F to $\tilde{\mathcal{G}}$. We assume that $S(\mathcal{G})$ consists of isolated points $\{p_1, \dots, p_k\}$. We set that, near p_j , $f_i^{(j)}$ are

the coefficients of $F^*\tilde{\omega}_\alpha^{(j)}$ such that we write $F^*\omega_\alpha^{(j)} = f_1^{(j)}dx_1 + \dots + f_n^{(j)}dx_n$. Then we have

$$\begin{aligned} \int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) &= \sum_{l=0}^n \int_X c_{n-l}(\Theta_X) c_1(\mathcal{G})^l \\ &= \sum_{j=1}^k \text{Res}_{p_j} \left[\frac{df_1^{(j)} \wedge \dots \wedge df_n^{(j)}}{f_1^{(j)} \dots f_n^{(j)}} \right]. \end{aligned}$$

Now we have the result.

Theorem 5.1 (Residue formula for non-transversality). *Let $F : X^n \rightarrow Y^m$ be a holomorphic map of generic rank r and $\tilde{\mathcal{G}}$ a codimension 1 non-singular distribution of Y . We assume that the non-transversal points of F to $\tilde{\mathcal{G}}$ are $\{p_1, \dots, p_k\}$, then we have*

$$\chi(X) + \sum_{l=1}^r \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathcal{G}})^l = \sum_{j=1}^k \text{Res}_{p_j} \left[\frac{df_1^{(j)} \wedge \dots \wedge df_n^{(j)}}{f_1^{(j)} \dots f_n^{(j)}} \right].$$

Proof. We denote by X^* the set of generic points where F has rank k . By using projection formula,

$$\begin{aligned} \int_X c_{n-l}(\Theta_X) c_1(\mathcal{G})^l &= \int_{X^*} c_{n-l}(\Theta_X) F^*(c_1(\tilde{\mathcal{G}}))^l \\ &= \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathcal{G}})^l. \end{aligned}$$

It is obvious that the above terms are zero for $k \leq l$.

Here let $F : X^2 \rightarrow Y^m$ be a map from compact complex surface. In this case we write down the above general form of the formula into geometric forms. We set that $y_m = F_m^{(j)}(x_1, x_2)$ is the m -th entry of a local representation of F near p_j and also write $dF_m^{(j)} = f_1^{(j)}dx_1 + f_2^{(j)}dx_2$. Then the above formula is

$$\chi(X) + \int_{F_*(c_1(X) \frown [X])} c_1(\tilde{\mathcal{G}}) + \int_{F_*[X]} c_1(\tilde{\mathcal{G}})^2 = \sum_{j=1}^k \text{Res}_{p_j} \left[\frac{df_1^{(j)} \wedge f_2^{(j)}}{f_1^{(j)}, f_2^{(j)}} \right].$$

We remark that if the generic rank of F is 1, the last term in the left-hand side of the above vanishes and we have

$$\chi(X) + \chi(M_F) \int_{F_*[X]} c_1(\tilde{\mathcal{G}}) = \sum_{j=1}^k \text{Res}_{p_j} \left[\frac{df_1^{(j)} \wedge f_2^{(j)}}{f_1^{(j)}, f_2^{(j)}} \right].$$

In the above, M_F is the generic fiber of F .

As an other example let us consider the case that $F : X^n \rightarrow C$ is a map for a curve C and $\tilde{\mathcal{G}} = \Omega_C$ is the point distribution. Then the above formula implies the multiplicity formula. (See [IS], [F].)

Theorem 5.2 (The multiplicity formula). *Let $F : X^n \rightarrow C$ be a holomorphic function for a compact complex curve C with the generic fiber M_F . If F has finite*

number of isolated critical points $\{p_1, \dots, p_k\}$, then we have

$$\chi(X) - \chi(M_F)\chi(C) = (-1)^n \sum_{j=1}^k \mu(F, p_j)$$

where $\mu(F, p_j)$ is the Milnor number of F at p_j .

Remarks. The one dimensional cases of theorem 4.1 is the classical Riemann-Hurwitz formula for a morphism of Riemann surfaces $F : C \rightarrow \tilde{C}$. We note that it cannot be deduced from the Baum-Bott type formula for $c_1(\mathcal{N}_{\mathcal{F}})$ in the above settings, however we can still apply the residue formula for \mathcal{G} in theorem 2.4. By taking the annihilator of the inverse image \mathcal{G} of $\Omega_{\tilde{C}}$, the given tangent sheaf \mathcal{F} of the lifted foliation turn out to be reduced. Since 1 dimensional manifolds only admits point foliations, the zero schemes of singularities are the points with multiplicities. Thus those kinds of singularities become non-singular by taking reduction. Therefore in our pull-back situation, the normal sheaf $\mathcal{N}_{\mathcal{F}}$ is always locally free and only \mathcal{G} itself keeps the informations of singularities of F .

REFERENCES

- [BB] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Diff. Geom. 7(1972), pp, 279-342.
- [F] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.
- [FG] W. Fulton and S. Lang, *Riemann-Roch Algebra*, Springer-Verlag, 1985.
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.
- [H] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.
- [IS] T. Izawa and T. Suwa, *Multiplicity of functions on singular varieties*, Internat. J. Math 14, No.1(2003), pp, 541-558.
- [KS] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer-Verlag, 1994.
- [L] D. Lehmann, *Systèmes d'alvéoles et intégration sur le complexe Čech-de Rham*, Publications de l'IRMA, 23, N VI, Université de Lille I, 1991.
- [O] P. Orlik, *The multiplicity of a holomorphic map at an isolated critical point*, Real and Complex Singularities, Oslo, 1979(Edited by P. Holm), Sitjhoff and Nordhoff, 1977.
- [S1] T. Suwa, *Residues of complex analytic foliation singularities*, J. Math. Soc. Japan. 36(1984), pp, 37-45.
- [S3] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann, Paris, 1998.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HOKKAIDO UNIVERSITY, SAPPORO 060, JAPAN

E-mail address: t-izawa@math.sci.hokudai.ac.jp